

Rotating thin-shell wormhole from glued Kerr spacetimes

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We construct a model of a rotating wormhole made by cutting and pasting two Kerr spacetimes. As a result, we obtain a rotating thin-shell wormhole with exotic matter at the throat. Two candidates for the exotic matter are considered: (i) a perfect fluid; (ii) an anisotropic fluid. We show that a perfect fluid is unable to support a rotating thin-shell wormhole. On the contrary, the anisotropic fluid with the negative energy density can be a source for such a geometry.

1 Introduction

Wormholes are usually defined as topological handles in spacetime linking widely separated regions of a single universe, or “bridges” joining two different spacetimes [1]. Their history traces back to the works of Einstein and Rosen [2], and Misner and Wheeler [3]. The modern interest in wormholes dates back to 1988, when Morris and Thorne [4] discussed the possibility of using wormholes for interstellar travels. As is well-known [4, 5], traversable wormholes can exist only if their throats contain exotic matter which possesses a negative pressure and violates the null energy condition. The search for realistic physical models providing the wormhole existence represents an important direction in wormhole physics. In general relativity there are models of wormholes supported by matter with exotic equations of state such as phantom energy [6, 7], a Chaplygin gas [8], tachyon matter [9]. Numerous examples of wormhole solutions have been found in various modifications of general relativity such as scalar-tensor theories of gravity, brane theories, semiclassical gravity, theories with non-minimal coupling [10, 11]. It is worth being noticed that most of the investigations deal with static spherically symmetric wormholes because of their

simplicity and high symmetry. At the same time, it would be important and interesting from a physical point of view to study wider classes of wormholes including non-static and rotating ones.

Rotating wormholes were first considered by Teo [12] who discussed some general geometrical properties of the stationary rotating wormhole spacetime. Other investigations in this field include studies of general requirements to the stress-energy tensor necessary to generate a rotating wormhole [13], energy conditions in a rotating wormhole spacetime and its traversability [14], and scalar perturbations in the rotating wormhole background [15]. Arguments in favor of the possibility of existence of semiclassical rotating wormholes were given in [16]. Solutions describing slowly rotating wormholes have been found and analyzed in [17, 18]. A number of new axially symmetric stationary exact solutions in general relativity with phantom and Maxwell fields have recently been obtained in [19, 20]; among them are solutions which represent rotating and magnetized wormholes.

The first examples of thin-shell wormholes have been given by Visser [21, 22]. In particular, he considered a spherically symmetric thin-shell wormhole constructed by joining two Schwarzschild geometries [22]. Generally, thin-shell wormholes are made by cutting and pasting two manifolds to form a geodesically complete new one with a throat lo-

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cated on the joining shell. In this case, the exotic matter needed to build the wormhole is concentrated on the shell, and the junction-condition formalism is used for its study. Due to elegance and relative simplicity, the cut-and-paste approach has become generally used for constructing new models of thin-shell wormholes such as charged wormholes [23], those with a cosmological constant [24], cylindrical [25] and plane-symmetric [26] wormholes, those in dilaton [27], Einstein–Gauss–Bonnet [28, 29], and Brans–Dicke [30] gravity, wormholes with a generalized Chaplygin gas [31], wormholes associated with global cosmic strings [32] and global monopoles [33]. Worth mentioning is also the paper by Bronnikov and Starobinsky [34] who considered static, spherically symmetric thin-shell wormholes in any non-ghost scalar-tensor theory of gravity and showed that the shell surface energy density is negative in all such cases.

In this paper we will apply the cut-and-paste method in order to construct and study a rotating thin-shell wormhole made by cutting and pasting two Kerr spacetimes.

2 Kerr surgery

The Kerr metric in the Boyer–Lindquist coordinates reads [35]

$$ds^2 = \left(1 - \frac{2mr}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left(r^2 + a^2 + \frac{2ma^2r}{\rho^2} \sin^2 \theta\right) \sin^2 \theta d\phi^2 + \frac{4mar}{\rho^2} \sin^2 \theta d\phi dt, \quad (1)$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 - 2mr + a^2$. The parameters m and $J = ma$ correspond to the mass and angular momentum of a Kerr black hole measured by a distant observer. The metric (1) has two fictitious singularities. The first one occurs at the *event horizon* $r = r_+$ where $\Delta = 0$, and hence g_{rr} is infinite:

$$r_+ = m + \sqrt{m^2 - a^2}. \quad (2)$$

The second singularity occurs on the boundary of the *ergosphere* $r = r_0$ where $g_{tt} = 0$:

$$r_0 = m + \sqrt{m^2 - a^2 \cos^2 \theta}. \quad (3)$$

Consider two copies \mathcal{M}_1 and \mathcal{M}_2 of the region $r \geq b$ of the Kerr spacetime (1):

$$\mathcal{M}_{1,2} = \{(t, r, \theta, \phi) \mid r \geq b\}. \quad (4)$$

As a result, we get two geodesically incomplete manifolds with boundaries given by the timelike hypersurfaces

$$\Sigma_{1,2} = \{(t, r, \theta, \phi) \mid F(r) = r - b = 0\}. \quad (5)$$

Identifying these hypersurfaces (i.e., $\Sigma = \Sigma_1 \equiv \Sigma_2$), we obtain a new manifold $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$, which is geodesically complete and possesses two asymptotically flat regions connected by a wormhole with the throat Σ . Note that the two-dimensional surface $t = \text{const}$, $r = b$ in Kerr spacetime is actually an ellipsoid of revolution having minor and major axes equal to b and $2(a^2 + b^2)^{1/2}$, respectively. Nevertheless, for brevity we will call b the wormhole throat radius. To avoid the presence of horizons in the resulting manifold \mathcal{M} , we will suppose $b > r_+$. Since \mathcal{M} is piecewise Kerr, the stress-energy tensor is everywhere zero, except for the throat itself. At Σ one may expect a stress-energy tensor proportional to the delta function. This means that the throat Σ is a thin shell.

To analyze such a thin-shell configuration, we will follow the Darmois–Israel standard formalism [36], also known as the junction condition formalism. The wormhole throat Σ is a synchronous timelike hypersurface, where we define the intrinsic coordinates $\xi^i = (\tau, \vartheta, \varphi)$ as follows: $\tau = t_1 \equiv t_2$, $\vartheta = \theta_1 \equiv \pi - \theta_2$, and $\varphi = \phi_1 \equiv \phi_2$. The coordinate τ is the proper time on the shell. Generally, the throat radius can be a function of proper time. However, we will assume $b(\tau) \equiv b = \text{const}$. Note that the metric (the first fundamental form) is continuous on Σ :

$$g_{ij}^1|_{\Sigma} = g_{ij}^2|_{\Sigma}, \quad (6)$$

while its first derivatives can be discontinuous. To describe this discontinuity, one should consider the extrinsic curvature. The extrinsic curvatures (second fundamental forms) associated with the two sides of the shell Σ are

$$K_{ij}^{\pm} = -n_{\gamma}^{\pm} \left(\frac{\partial^2 x^{\gamma}}{\partial \xi^i \partial \xi^j} + \Gamma_{\alpha\beta}^{\gamma} \frac{\partial x^{\alpha}}{\partial \xi^i} \frac{\partial x^{\beta}}{\partial \xi^j} \right) \Big|_{\Sigma}, \quad (7)$$

where n_{γ}^{\pm} are the unit normals ($n^{\gamma} n_{\gamma} = 1$) to Σ :

$$n_{\gamma}^{\pm} = \pm \left| g^{\alpha\beta} \frac{\partial F}{\partial x^{\alpha}} \frac{\partial F}{\partial x^{\beta}} \right|^{-1/2} \frac{\partial F}{\partial x^{\gamma}}. \quad (8)$$

Generally, $K_{ij}^+ \neq K_{ij}^-$. With the definitions $k_{ij} = K_{ij}^+ - K_{ij}^-$ and $k = k_i^i$ we have the Einstein equations on the shell (also called the Lanczos equations)

$$-k_{ij} + kg_{ij} = 8\pi S_{ij}, \quad (9)$$

where S_{ij} is the surface stress-energy tensor.

Let us adopt the orthonormal basis $\{\mathbf{e}_{\hat{r}}, \mathbf{e}_{\hat{\theta}}, \mathbf{e}_{\hat{\varphi}}\}$ for the metric (1) on Σ :

$$\begin{aligned} \mathbf{e}_{\hat{r}} &= \frac{\mathbf{e}_r - \frac{g_{r\varphi}}{g_{\varphi\varphi}}\mathbf{e}_\varphi}{\sqrt{g_{rr} - \frac{g_{r\varphi}^2}{g_{\varphi\varphi}}}}, \\ \mathbf{e}_{\hat{\theta}} &= \frac{\mathbf{e}_\theta}{\sqrt{-g_{\theta\theta}}}, \\ \mathbf{e}_{\hat{\varphi}} &= \frac{\mathbf{e}_\varphi}{\sqrt{-g_{\varphi\varphi}}}. \end{aligned} \quad (10)$$

In this basis, the surface stress-energy tensor S_{ij} has the following algebraic structure:

$$S_{ij} = \begin{bmatrix} \sigma & 0 & \zeta \\ 0 & p_\theta & 0 \\ \zeta & 0 & p_\varphi \end{bmatrix}, \quad (11)$$

where σ is the surface energy density, p_θ and p_φ are the principal surface pressures, and ζ is the surface angular momentum density. The Lanczos equations (9) in the basis (10) take the following form:

$$4\pi\sigma = -\frac{\Delta_\beta^{1/2}}{m\rho_\beta\Phi} \left[2\beta^3 + \alpha^2\beta + \alpha^2 + \alpha^2(\beta - 1)\cos^2\vartheta \right], \quad (12a)$$

$$4\pi p_\theta = \frac{\beta - 1}{m\rho_\beta\Delta_\beta^{1/2}}, \quad (12b)$$

$$\begin{aligned} 4\pi p_\varphi &= \frac{1}{m\rho_\beta^3\Delta_\beta^{1/2}\Phi} \left[\beta^2(\beta^5 - \beta^4 + 2\alpha^2\beta^3 \right. \\ &\quad + 2\alpha^2\beta^2 + \alpha^2\beta(\alpha^2 - 8) + 3\alpha^4) \\ &\quad + \alpha^2\cos^2\vartheta(\beta^5 - 5\beta^4 + 2\beta^3(\alpha^2 + 4) \\ &\quad \left. - 6\alpha^2\beta^2 + \alpha^4\beta - \alpha^4) \right], \end{aligned} \quad (12c)$$

$$4\pi\zeta = -\frac{1}{m\rho_\beta^3\Phi} \left[\alpha\sin\vartheta(3\beta^4 + \alpha^2\beta^2 + \alpha^2(\beta^2 - \alpha^2)\cos^2\vartheta) \right], \quad (12d)$$

where we have introduced the convenient dimensionless quantities

$$\beta = bm^{-1}, \quad \alpha = am^{-1},$$

$$\Delta_\beta = \beta^2 - 2\beta + \alpha^2, \quad \rho_\beta^2 = \beta^2 + \alpha^2\cos^2\theta,$$

$$\Phi = \beta^4 + \alpha^2\beta^2 + 2\alpha^2\beta + \alpha^2\Delta_\beta\cos^2\vartheta.$$

Later on we will also use dimensionless notations for the event horizon $\beta_+ = r_+m^{-1} = 1 + \sqrt{1 - \alpha^2}$ and the boundary of ergosphere $\beta_0 = r_0m^{-1} = 1 + \sqrt{1 - \alpha^2\cos^2\theta}$.

3 Matter on the shell

It is necessary to emphasize that the quantities σ , p_θ , p_φ , and ζ given by Eqs. (12) are not yet related to any physical model of matter filling the shell Σ . Their values are of purely geometric nature and depend on the metric parameters m and a and the throat radius b . To impart a physical sense to these quantities one should specify the kind of matter which can support the rotating thin-shell wormhole.

3.1 Perfect fluid

As a simple model of matter located on the shell Σ , we will first consider a perfect fluid. In the orthonormal basis (10) the surface stress-energy tensor of a perfect fluid is

$$S_{ij} = (\mathcal{E} + \mathcal{P})u_iu_j - \eta_{ij}\mathcal{P}, \quad (13)$$

where $\eta_{ij} = \text{diag}(+1, -1, -1)$, u_i is the fluid velocity which is supposed to be timelike, i.e. $u^i u_i = 1$, \mathcal{E} is the fluid energy density measured in the co-moving frame, and \mathcal{P} is the pressure isotropic in all directions tangent to the shell Σ . For the rotating fluid it is naturally to choose $u_i = (u_\tau, 0, u_\varphi)$. Comparing (11) and (13), we find

$$\sigma = (\mathcal{E} + \mathcal{P})u_\tau^2 - \mathcal{P}, \quad (14a)$$

$$p_\theta = \mathcal{P}, \quad (14b)$$

$$p_\varphi = (\mathcal{E} + \mathcal{P})u_\varphi^2 + \mathcal{P}, \quad (14c)$$

$$\zeta = (\mathcal{E} + \mathcal{P})u_\tau u_\varphi. \quad (14d)$$

Combining these equations, one can easily obtain the following relation:

$$(\sigma + p_\theta)(p_\varphi - p_\theta) - \zeta^2 \equiv 0. \quad (15)$$

Substitution of Eqs. (12) into the last relation gives

$$4\alpha^2\beta^2\rho_\beta^{-6}\sin^2\theta \equiv 0. \quad (16)$$

This identity is only fulfilled provided $\alpha = am^{-1} = 0$, i.e., $a = 0$. Therefore, a perfect fluid cannot be a source for a rotating thin-shell wormhole with $a \neq 0$.²

3.2 Anisotropic fluid

Now consider an anisotropic fluid with the surface stress-energy tensor

$$S_{ij} = \mathcal{E}u_i u_j + \mathcal{P}_1 v_i v_j + \mathcal{P}_2 \Pi_{ij}, \quad (17)$$

Here $u_i = (u_\tau, 0, u_\varphi)$ is the fluid timelike velocity ($u^i u_i = 1$), and v_i and Π_{ij} satisfy the following orthogonality conditions:

$$u^i v_i = 0, \quad u^i \Pi_{ij} = 0, \quad v^i \Pi_{ij} = 0. \quad (18)$$

\mathcal{E} is the energy density, \mathcal{P}_1 and \mathcal{P}_2 are the fluid pressures in two orthogonal directions tangent to the shell Σ (generally, $\mathcal{P}_1 \neq \mathcal{P}_2$). For the rotating fluid it is natural to choose $u_i = (u_\tau, 0, u_\varphi)$ with

$$u_\tau^2 - u_\varphi^2 = 1, \quad (19)$$

and $v_i = (0, 1, 0)$; the tensor Π_{ij} can be constructed as follows: $\Pi_{ij} = u_i u_j - v_i v_j - \eta_{ij}$. Comparing (11) and (17), we find

$$\sigma = (\mathcal{E} + \mathcal{P}_2)u_\tau^2 - \mathcal{P}_2, \quad (20a)$$

$$p_\vartheta = \mathcal{P}_1, \quad (20b)$$

$$p_\varphi = (\mathcal{E} + \mathcal{P}_2)u_\varphi^2 + \mathcal{P}_2, \quad (20c)$$

$$\zeta = (\mathcal{E} + \mathcal{P}_2)u_\tau u_\varphi, \quad (20d)$$

The latter equations, together with the normalizing condition (19), form a set of five algebraic equations for five unknowns \mathcal{E} , \mathcal{P}_1 , \mathcal{P}_2 , u_τ and u_φ . Resolving the system yields $\mathcal{P}_1 = p_\vartheta$, and

$$\mathcal{E}^\pm = \frac{1}{2} \left[\sigma - p_\varphi \pm \sqrt{D} \right], \quad (21a)$$

$$\mathcal{P}_2^\pm = \frac{1}{2} \left[-\sigma + p_\varphi \pm \sqrt{D} \right], \quad (21b)$$

$$(u_\tau^\pm)^2 = \pm \frac{\sigma + p_\varphi}{2\sqrt{D}} + \frac{1}{2}, \quad (21c)$$

$$(u_\varphi^\pm)^2 = \pm \frac{\sigma + p_\varphi}{2\sqrt{D}} - \frac{1}{2}, \quad (21d)$$

with $D = (\sigma + p_\varphi)^2 - 4\zeta^2$. It is worth noting that we have got two classes of solutions which depend

on a choice of the plus or minus sign in the obtained expressions.

Finally, Eqs. (21) represent expressions for the surface energy density \mathcal{E} , pressures \mathcal{P}_1 and \mathcal{P}_2 , and velocity components u_τ and u_φ of the anisotropic fluid on the shell Σ .

4 Analysis

In this section we will analyze the model of a rotating thin-shell wormhole constructed above. First of all, let us consider the particular case of a non-rotating thin-shell wormhole with $a = 0$ (no angular momentum). In this case the metric (1) reduces to the Schwarzschild one, and Eqs. (12) reduce to those obtained by Visser [22]:

$$\begin{aligned} \sigma &= -\frac{1}{2\pi b} \sqrt{1 - 2m/b}, \\ p_\vartheta = p_\varphi &= \frac{1}{4\pi b} \frac{1 - m/b}{\sqrt{1 - 2m/b}}, \quad \zeta = 0. \end{aligned} \quad (22)$$

Note that the surface energy density σ tends to zero and the pressures p_ϑ and p_φ to infinity if the throat radius b tends to that of the event horizon $r_g = 2m$.

In the general case of a rotating thin-shell wormhole with $a \neq 0$ we have $\sigma \sim \Delta_\beta^{1/2}$ and $p_\vartheta, p_\varphi \sim \Delta_\beta^{-1/2}$ (see (12)). Since $\Delta_\beta = 0$ if $\beta = \beta_+ := 1 + \sqrt{1 - \alpha^2}$, we can see that $\sigma \rightarrow 0$ and $p_\vartheta, p_\varphi \rightarrow \infty$ as $\beta \rightarrow \beta_+$.

Now let us discuss the properties of the anisotropic fluid located on the shell Σ . Given the expressions (12) for σ , p_ϑ , p_φ , and ζ , we can find the values \mathcal{E} , \mathcal{P}_1 , \mathcal{P}_2 , u_τ , and u_φ as explicit functions of the dimensionless throat radius β . In particular, we have

$$\begin{aligned} D &= \frac{1}{4\pi m \rho_\beta^6 \Delta_\beta} \left[\beta^3 (\beta(\beta - 3)^2 - 4\alpha^2) \right. \\ &\quad \left. + 2\alpha^2 \beta \cos^2 \vartheta (\beta^3 - 3\beta + 2\alpha^2) + \alpha^4 \cos^4 \vartheta (\beta - 1)^2 \right]. \end{aligned} \quad (23)$$

Note that D should necessarily be positive, i.e., $D > 0$. As is shown in the Appendix, it is possible if and only if $\beta \in I_1 \cup I_2$, where $I_1 = (\beta_+, \beta_2)$, $I_2 = (\beta_3, \infty)$, and

$$\beta_n = 2 + 2 \cos \left(\frac{\chi - 2\pi(3 - n)}{3} \right), \quad n = 1, 2, 3,$$

²Nevertheless, a perfect fluid can support a spherically symmetric thin-shell wormhole made of two surgically modified Schwarzschild spacetimes [22].

(24)

with χ defined by $\cos \chi = 2\alpha^2 - 1$. Additionally, one should check whether or not the values of $(u_\tau^\pm)^2$ and $(u_\varphi^\pm)^2$ given by Eqs. (21c) and (21d) are non-negative.³ From Fig. 1 one may see that $(u_\tau^+)^2$ and $(u_\varphi^+)^2$ are positive if $\beta < \beta_2$, while $(u_\tau^-)^2$ and $(u_\varphi^-)^2$ are positive if $\beta > \beta_3$. This means that one should take the plus sign in Eqs. (21a)–(21d) in the case $\beta \in I_1$ and the minus sign if $\beta \in I_2$. Let us repeat that the domain $\beta \leq \beta_+$ is forbidden by definition since we consider only wormholes whose throat radius is greater than that of the event horizon β_+ . In addition, it turns out that the domain $\beta \in [\beta_2, \beta_3]$ is also forbidden for rotating thin-shell wormholes. Thus we have two classes of wormhole solutions depending on the throat radius β : (i) $\beta_+ < \beta < \beta_2$; (ii) $\beta > \beta_3$.

The energy density \mathcal{E} and the pressures \mathcal{P}_1 and \mathcal{P}_2 as functions of β are shown in Fig. 2. Note that \mathcal{E} is negative, while \mathcal{P}_1 and \mathcal{P}_2 are positive for all values of β .

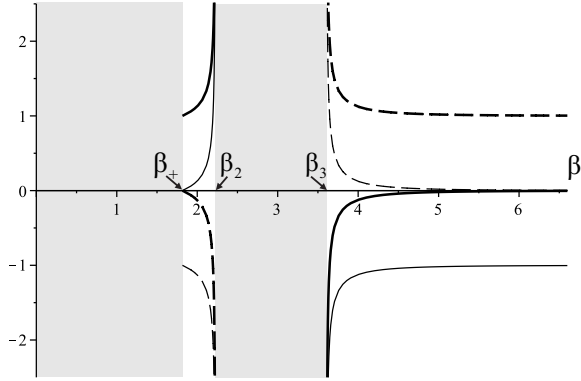


Figure 1: Plots of $(u_\tau^\pm)^2$ and $(u_\varphi^\pm)^2$ vs. β with given $\alpha = 0.5$, $m = (4\pi)^{-1}$. The solid and dashed curves are used for the plus- and minus-sign solutions, respectively; thick lines show $(u_\tau^\pm)^2$, and thin lines show $(u_\varphi^\pm)^2$. The shaded areas indicate forbidden regions $\beta \leq \beta_+$ and $\beta \in [\beta_2, \beta_3]$.

³In principle, one may discard this requirement and consider also negative values of u_τ^2 and u_φ^2 . In this case the components u_τ and u_φ will be pure imaginary, and as a consequence u^i will be spacelike, i.e. $u^i u_i = -1$. In turn, this means that the fluid velocity exceeds the velocity of light.

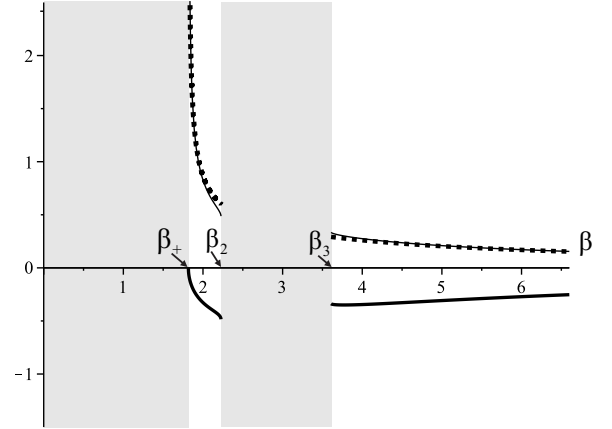


Figure 2: Plots of \mathcal{E} , \mathcal{P}_1 , and \mathcal{P}_2 vs. β with given $\alpha = 0.5$, $m = (4\pi)^{-1}$. Solid, dotted, and thick lines show \mathcal{E} , \mathcal{P}_1 , and \mathcal{P}_2 , respectively. The shaded areas indicate forbidden regions $\beta \leq \beta_+$ and $\beta \in [\beta_2, \beta_3]$.

5 Conclusion

We have constructed a rotating wormhole model by cutting and pasting two Kerr spacetimes. As is usual for the cut-and-paste approach, the resulting wormhole spacetime has a thin shell joining two regions of Kerr spacetimes. This shell represents the wormhole throat and contains exotic matter needed to support the wormhole. We have discussed two possible candidates to the role of the exotic matter: (i) a perfect fluid, and (ii) an anisotropic fluid. It has been shown that a perfect fluid is unable to support a rotating thin-shell wormhole, while an anisotropic fluid localized on the shell can be a source of such geometry. The corresponding fluid energy density \mathcal{E} and anisotropic pressures \mathcal{P}_1 and \mathcal{P}_2 are given by Eqs. (21) which express \mathcal{E} , \mathcal{P}_1 , and \mathcal{P}_2 as functions of the dimensionless throat radius β . Admissible values of β belong to two nonintersecting intervals $I_1 = (\beta_+, \beta_2)$ and $I_2 = (\beta_3, \infty)$, where $\beta_+ = 1 + \sqrt{1 - \alpha^2}$ is the event horizon, and β_n ($n = 2, 3$) are given by Eq. (24). Since $\beta_2 < \beta_3$, the intervals I_1 and I_2 are not intersected. Therefore, there are two classes of wormhole solutions: (i) with “small” throat radii $\beta_+ < \beta < \beta_2$, and (ii) with “large” radii $\beta > \beta_3$. In both cases the energy density \mathcal{E} of the anisotropic fluid turns out to be negative. This means that matter supporting the rotating wormhole violates the weak energy condition.

It is interesting that the throat radius β of the rotating thin-shell wormhole can be less than the maximal size of ergosphere $\beta_0^{max} = 2$ ($\theta = \pi/2$). This is possible for wormholes of the class I with small throat radii $\beta_+ < \beta < \beta_2$ (see the appendix). Moreover, for wormholes with large angular momentum $\alpha > 2^{-1/2}$ all values of β from the interval (β_+, β_2) are less than β_2^{max} . Thus, there are wormholes (of the class I) whose throat lies inside of the ergosphere. Such the feature may, in principle, lead to interesting consequences due to processes similar to the Penrose process in the ergosphere of Kerr black hole.

An important issue in wormhole physics is the stability of wormhole configurations. The stability of spherically symmetric thin-shell wormholes has been intensively considered in the literature [37–44]. We intend to study this problem for rotating thin-shell wormholes in our forthcoming paper.

Appendix

Rearranging Eqs. (20) yields

$$\zeta^2 = (\sigma + \mathcal{P}_2)(p_\varphi - \mathcal{P}_2), \quad (\text{A.1})$$

It is a quadratic equation for \mathcal{P}_2 with the discriminant $D = (\sigma + p_\varphi)^2 - 4\zeta^2$ which should be necessarily positive, $D > 0$. Using the relations (12a), (12c), and (12d), we find

$$\begin{aligned} D = (4\pi m)^{-1} \rho_0^{-6} \Delta_0^{-1} & \left[\beta^3(\beta(\beta - 3)^2 - 4\alpha^2) \right. \\ & + 2\alpha^2 \beta \cos^2 \vartheta (\beta^3 - 3\beta + 2\alpha^2) \\ & \left. + \alpha^4 \cos^4 \vartheta (\beta - 1)^2 \right]. \end{aligned} \quad (\text{A.2})$$

Since $b > r_+$ is assumed, we have $\beta > \beta_+ = 1 + \sqrt{1 - \alpha^2}$, and one may check in a straightforward manner that the cosine terms in (A.2) are positive. Therefore the sign of D is determined by the first term in the square brackets. In particular, on the equator $\vartheta = \pi/2$ the condition $D > 0$ reduces to

$$f_\alpha(\beta) = \beta(\beta - 3)^2 - 4\alpha^2 > 0. \quad (\text{A.3})$$

The cubic parabola $f_\alpha(\beta)$ has three roots β_n ($n = 1, 2, 3$) given by Cardano's formulas:

$$\beta_n = 2 + 2 \cos \left(\frac{\chi - 2\pi(3 - n)}{3} \right), \quad (\text{A.4})$$

with χ defined by

$$\cos \chi = 2\alpha^2 - 1.$$

In the case $0 < \alpha < 1$ all roots are real and different, such that $\beta_1 < \beta_2 < \beta_3$; if $\alpha = 0$, then $\beta_1 = 0$ and $\beta_2 = \beta_3 = 3$; if $\alpha = 1$, then $\beta_1 = \beta_2 = 1$ and $\beta_3 = 4$ (see Fig. 3). Formally, one can also consider $\alpha > 1$ (i.e., $a > m$); in this case β_1 and β_2 become imaginary, and β_3 is an only real root. In general, the solution of the inequality (A.3) reads

$$\beta \in (\beta_1, \beta_2) \cup (\beta_3, \infty).$$

In addition, let us recall that it is assumed $b > r_+$, hence $\beta > \beta_+ = 1 + \sqrt{1 - \alpha^2}$. One can check that $\beta_1 < \beta_+ < \beta_2$, and so we finally have

$$\beta \in (\beta_+, \beta_2) \cup (\beta_3, \infty). \quad (\text{A.5})$$

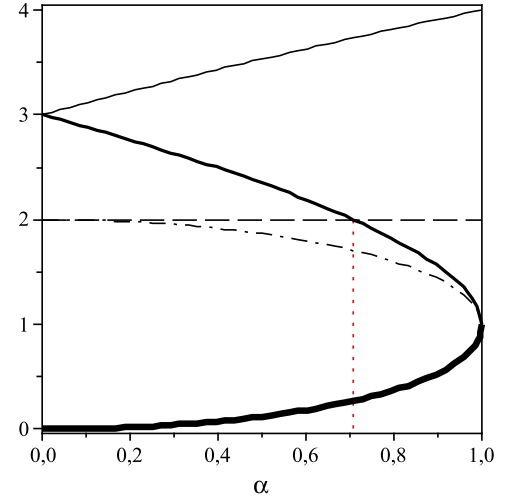


Figure 3: Graphs of roots β_n vs. α . Thick, middle, and thin lines denote β_1 , β_2 , and β_3 , respectively. The dot-dashed line indicates the event horizon $\beta_+ = 1 + \sqrt{1 - \alpha^2}$. The dashed line shows the maximal size of ergosphere $\beta_0^{max} = 2$ ($\theta = \pi/2$). The lines for β_2 and β_0^{max} are intersected at $\alpha = 2^{-1/2}$.

Thus, admissible values of β belong to two nonintersecting intervals $I_1 = (\beta_+, \beta_2)$ and $I_2 = (\beta_3, \infty)$. Note that they can only be intersected if $\alpha = 0$ (no rotation), when $\beta_2 = \beta_3 = 3$. In this case one may obtain static, spherically symmetric thin-shell wormhole with the throat's radius $\beta = 3$, or $b = 3m$, whose value lies on the boundary between I_1 and I_2 [22].

It is also worth emphasizing that an admissible value of β can be less than the maximal size of ergosphere $\beta_0^{max} = 2$ ($\theta = \pi/2$). Really, in case $\beta \in (\beta_+, \beta_2)$ one may always choose $\beta_+ < \beta < \min(\beta_0^{max}, \beta_2)$ (see Fig. 3). Moreover, for $\alpha > 2^{-1/2}$ one has $\beta_2 < 2$, hence all values of β from the interval (β_+, β_2) are less than β_2^{max} .

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